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LETTER TO THE EDITOR

On conformally covariant spin-2 and spin-$\frac{3}{2}$ equations

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Abstract. The standard massless spin-2 and spin-$\frac{3}{2}$ equations are not conformally covariant. By varying the coefficients of various terms in these equations we derive conformally covariant equations. These equations are then used to construct consistent coupling terms in the original field equations representing the self-interaction and the interaction of massless spin-2 (or $\frac{3}{2}$) fields with matter.

The theory of massless spin-2 tensor fields to describe the gravitational field has a long history. In particular, there are longstanding difficulties concerning the coupling of the massless spin-2 field to a matter field in the Minkowski space (Wyss 1965).

We wish to clarify in this letter several points regarding the conformal covariance of both massless spin-2 and spin-$\frac{3}{2}$ fields and their coupling to matter. The standard field equations for these fields are by themselves not conformally covariant (Bracken and Jessup 1981 (this paper contains a very comprehensive treatment of conformal covariance of all massless equations)), contrary to some general beliefs that all massless equations are conformally covariant. We show here that they can be made so, if suitably combined with spin-0 and spin-$\frac{1}{2}$ fields, respectively. We then use these conformally covariant Lagrangians to construct consistent field equations for spin-2 and spin-$\frac{3}{2}$ fields coupled to the combinations of spin-0 and -2 (or spin-$\frac{1}{2}$ and -$\frac{3}{2}$) fields, representing *self-coupling* plus coupling to matter. The reasons for these mixing lie in the representation theory of the conformal group. Usually fields are classified by their transformation properties under the Lorentz group, and then one looks to see if the massless equations are invariant under the larger conformal group. If, however, one introduces the fields according to the finite-dimensional representations of the conformal group itself, then they contain, when reduced with respect to the Lorentz group, in general, several different spin fields.

The symmetric tensor $h_{\mu\nu} = h_{\nu\mu}$, defined by its Lorentz transformation property, transforms under special conformal transformations as (Isham *et al* 1970)

$$\begin{aligned} h'_{\mu\nu} &= \Omega D_{\mu\nu}{}^{\alpha\beta} h_{\alpha\beta} \\ &= \Omega[g_\mu{}^\alpha g_\nu{}^\beta + (c^\lambda x^\sigma - x^\lambda c^\sigma)(I_{\lambda\sigma})_{\mu\nu}{}^{\alpha\beta}]h_{\alpha\beta}, \end{aligned} \tag{1}$$

where

$$(I_{\lambda\sigma})_{\mu\nu}{}^{\alpha\beta} = (g_{\lambda\mu} g_\sigma{}^\alpha - g_\lambda{}^\alpha g_{\sigma\mu}) g_\nu{}^\beta + (g_{\lambda\nu} g_\sigma{}^\beta - g_\lambda{}^\beta g_{\sigma\nu}) g_\mu{}^\alpha$$

and

$$\Omega = 1 + 2c^\mu x_\mu + c^2 x^2. \tag{2}$$

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Equation (1) can also be written as

$$h'_{\mu\nu}(x') = \Omega\Lambda_\mu{}^\alpha(x)\Lambda_\nu{}^\beta(x)h_{\alpha\beta}(x),$$

where

$$\Lambda_\mu{}^\alpha(x) = g_\mu{}^\alpha + 2(c_\mu x^\alpha - x_\mu c^\alpha).$$

It is known that the Fierz–Pauli equation for this massless tensor field

$$G^{(\mathrm{FP})}_{\mu\nu} \equiv \partial^2 h_{\mu\nu} - \partial_\mu\partial^\sigma h_{\nu\sigma} - \partial_\nu\partial^\sigma h_{\mu\sigma} + \partial_\mu\partial_\nu h + g_{\mu\nu}(\partial^\lambda\partial^\sigma h_{\lambda\sigma} - \partial^2 h) = T_{\mu\nu} \tag{3}$$

is not conformally covariant, as can easily be verified. Equation (3) is however gauge invariant under the transformation

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu, \tag{4}$$

where ξ_μ is an arbitrary vector field. Consequently, equation (4) selects the spin-2 part of the tensor field. Furthermore, we have for equation (3)

$$\partial^\mu T_{\mu\nu} = 0, \tag{5}$$

because the left-hand side of (3) satisfies this equation automatically:

$$\partial^\mu G^{(\mathrm{FP})}_{\mu\nu} \equiv 0. \tag{6}$$

However, it is not possible to couple $h_{\mu\nu}$ to a matter field $T_{\mu\nu}$ of definite spin. For example, if we insert in (3) for $T_{\mu\nu}$ the energy–momentum tensor $T_{\mu\nu}(\phi)$ of a scalar field ϕ, equation (6) leads to an inconsistency, because for the coupled $(h_{\mu\nu} - \phi)$ fields, $\partial^\mu T_{\mu\nu}(\phi) \neq 0$ (although $\partial^\mu T_{\mu\nu}(\phi) = 0$ for a *free* scalar field). Successive higher-order coupling terms to eliminate this inconsistency order by order then leads to Einstein's Lagrangian (Wyss 1965).

The conformally covariant field equations for the tensor field can be obtained by readjusting the coefficients of the various terms in equation (3). In this way we derive the conformally covariant Lagrangian

$$\mathscr{L}^{\mathrm{c}}(h) = -\tfrac{1}{2}(\partial_\sigma h_{\mu\nu})^2 + \tfrac{2}{3}(\partial_\sigma h_{\mu\nu}\partial^\nu h^{\mu\sigma}) + \tfrac{1}{6}(\partial_\sigma h)^2 - \tfrac{1}{3}\partial_\sigma h\,\partial_\mu h^{\mu\sigma}, \tag{7}$$

where

$$h \equiv h^\mu{}_\mu, \tag{8}$$

and transforms as

$$h'_\mu{}^\mu = \Omega h_\mu{}^\mu.$$

Under special conformal transformations we find indeed that

$$\mathscr{L}'(h'_{\mu\nu}) = \Omega^4\mathscr{L}(h_{\mu\nu}) + 2c^\lambda R_\lambda, \qquad R_\lambda = \partial^\sigma R_{\sigma\lambda},$$
$$R_{\sigma\lambda} = \tfrac{1}{3}[\tfrac{1}{2}g_{\sigma\lambda}(h_{\mu\nu})^2 + 2(h_{\sigma\nu}h^\nu{}_\lambda) - (hh_{\sigma\lambda})] \tag{9}$$

and $\mathscr{L}'$ leads to the same field equations (Flato *et al* 1970).

The resultant equations of motion from (7) are

$$G^{(\mathrm{c})}_{\mu\nu} \equiv \partial^2 h_{\mu\nu} - \tfrac{2}{3}(\partial_\mu\partial^\sigma h_{\nu\sigma} + \partial_\nu\partial^\sigma h_{\mu\sigma}) + \tfrac{1}{3}\partial_\mu\partial_\nu h + \tfrac{1}{3}g_{\mu\nu}(\partial^\lambda\partial^\sigma h_{\lambda\sigma} - \partial^2 h) = 0. \tag{10}$$

Equation (10) was also given recently by Drew and Greenberg (1980). Equation (10) is conformally covariant, but not gauge invariant under (4). This implies that we have both spin-0 and spin-2 components of $h_{\mu\nu}$, and not a pure spin-2 field. However, $G^{(\mathrm{c})}_{\mu\nu}$

automatically satisfies

$$G^{(c)\mu}_{\mu} \equiv 0. \tag{11}$$

But $G^{(c)}_{\mu\nu}$ is not automatically conserved:

$$\partial^{\mu} G^{(c)}_{\mu\nu} \neq 0, \tag{12}$$

unless

$$\partial^{\lambda}\partial^{\rho}(\partial_{\rho} h_{\mu\lambda} - \partial_{\mu} h_{\lambda\rho}) = 0. \tag{13}$$

The first question is now whether we can couple this field equation $G^{(c)}_{\mu\nu}$ to a matter field in the form

$$G^{(c)}_{\mu\nu} = \theta_{\mu\nu}, \tag{14}$$

where $\theta_{\mu\nu}$ is the conformally covariant energy–momentum tensor of a matter field. Now the conformally covariant tensor $\theta_{\mu\nu}$ does indeed satisfy $\theta^{\mu}{}_{\mu} = 0$, but not necessarily $\theta^{\mu\nu}, \nu = 0$ *for interacting fields.* There are models of nonlinear conformally covariant matter field theories for scalar and spinor fields (Barut and Xu 1981a, b), for which $\theta^{\mu\nu}, \nu = 0$, but these can not be coupled to the tensor field $h_{\mu\nu}$.

We can obtain, however, a consistent system of coupled field equations as follows. If we add a term

$$T_{\mu\nu} = -\tfrac{1}{3}(\partial_{\mu}\partial^{\sigma} h_{\nu\sigma} + \partial_{\nu}\partial^{\sigma} h_{\mu\sigma}) + \tfrac{2}{3}\partial_{\mu}\partial_{\nu} h + \tfrac{2}{3} g_{\mu\nu}(\partial^{\lambda}\partial^{\sigma} h_{\lambda\sigma} - \partial^2 h) \tag{15}$$

to both sides of equation (10), we have

$$G^{(c)}_{\mu\nu} + T_{\mu\nu} = T_{\mu\nu} \tag{16}$$

or

$$G^{(FP)}_{\mu\nu} = T_{\mu\nu}, \tag{16'}$$

where the left-hand side is the Fierz–Pauli equation (3). Since $\partial^{\mu} G^{(FP)}_{\mu\nu} = 0$ by equation (6), we must impose the condition

$$\partial^{\mu} T_{\mu\nu} = 0, \tag{17}$$

which implies

$$\partial^{\sigma}(\partial_{\nu}\partial^{\mu} h_{\mu\sigma} - \partial^2 h_{\nu\sigma}) = 0, \tag{18}$$

which is exactly the same as equation (13). Thus, under the condition (18) we have a coupling of the Fierz–Pauli equation for $h_{\mu\nu}$ to itself and to $h \equiv h^{\mu}{}_{\mu} = \phi$, which behaves like a scalar field, with

$$T_{\mu\nu} = -\tfrac{1}{3}(\partial_{\mu}\partial^{\sigma} h_{\nu\sigma} + \partial_{\nu}\partial^{\sigma} h_{\mu\sigma}) + \tfrac{2}{3} g_{\mu\nu}\partial^{\lambda}\partial^{\sigma} h_{\lambda\sigma} + \tfrac{2}{3}(\partial_{\mu}\partial_{\nu} - g_{\mu\nu}\partial^2)\phi. \tag{19}$$

We now discuss the conformal invariance of massless spin-$\tfrac{3}{2}$–$\tfrac{1}{2}$ fields. This system is described by a field $\psi_{\mu} = \psi h_{\mu}$, where ψ is a Dirac spinor and h_{μ} a vector field, transforming under the special conformal transformations as (ψ_{μ} has the conformal weight $-\tfrac{3}{2}$)

$$\psi'_{\mu}(x') = \Omega(1 + \gamma^{\lambda}\gamma^{\sigma} c_{\lambda} x_{\sigma})\Lambda_{\mu}{}^{\rho}(x)\psi_{\rho}(x). \tag{20}$$

The Rarita–Schwinger equation (Schwinger 1970)

$$\gamma^{\lambda}\partial_{\lambda}\psi_{\mu} - (\partial_{\mu}\gamma^{\lambda}\psi_{\lambda} - \gamma_{\mu}\partial^{\lambda}\psi_{\lambda}) + \gamma_{\mu}\gamma^{\lambda}\partial_{\lambda}\gamma^{\sigma}\psi_{\sigma} = \eta_{\mu} \tag{21}$$

is gauge invariant under

$$\psi_\mu \to \psi h_\mu + \psi \partial_\mu \chi \tag{22}$$

so that one can select a pure spin-$\frac{3}{2}$ component, and

$$\partial^\mu \eta_\mu = 0. \tag{23}$$

The conformally covariant equation is again obtained by varying the coefficients of the terms in equation (21), and we get

$$\gamma^\lambda \partial_\lambda \psi_\mu - \tfrac{1}{2}(\partial_\mu \gamma^\lambda \psi_\lambda + \gamma_\mu \partial^\lambda \psi_\lambda) + \gamma_\mu \gamma^\lambda \partial_\lambda \gamma^\sigma \psi_\sigma = 0. \tag{24}$$

This is the desired conformally covariant equation which is, however, not gauge invariant.

As before, if we add the term

$$-\tfrac{1}{2}(\partial_\mu \gamma^\lambda \psi_\lambda + \gamma_\mu \partial^\lambda \psi_\lambda)$$

to both sides of equation (24), we obtain the Rarita–Schwinger equation (21) with a definite coupling term

$$\eta_\mu = -\tfrac{1}{2}(\partial_\mu \gamma^\lambda \psi_\lambda + \gamma_\mu \partial^\lambda \psi_\lambda) \tag{25}$$

and this includes both spin-$\frac{1}{2}$ and spin-$\frac{3}{2}$ components. The condition $\partial^\mu \eta_\mu = 0$ then implies

$$\partial^2 \gamma^\lambda \psi_\lambda + \gamma^\mu \partial_\mu \partial^\lambda \psi_\lambda = 0. \tag{26}$$

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